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Investigation of One-Step Methods for Integro-Differential Equations

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ABSTRACT

One-step methods for solving integro-differential equations are studied from the point of view of desiring that the method give good accuracy when the true solution asymptotically goes to zero. A test equation is proposed and absolute stability is investigated.

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1. INTRODUCTION

Behavior of a number of physical and biological systems may be described by integro-differential equations of the form

$$(1)\quad y'(x) = F\left(x, y(x), \int_0^x K(x, t, y(t))\, dt\right).$$

A familiar example is the population equation [2]

$$(2)\quad y'(x) = ay(x) - by^2(x) - y(x) \int_0^x K(x-t)\, y(t)\, dt,$$

in which y represents a population and x represents time. The integral factor takes into account "heredity", or in general the past history of the system. Thus in applications to mechanics such integral factors may provide for component fatigue.

In spite of their importance, only very recently has work been done on the development of efficient and accurate numerical methods for solving these equations. The simplest methods proposed [1, 3] are the one-step methods, analogous to Euler's method and its generalizations, in which the left hand side of (1) is replaced by a first-order difference $\frac{y(x+h) - y(x)}{h}$ and the right side is evaluated at x (or, more generally, is some convex linear combination of evaluations at x and x+h). Beginning with a given initial value y(0), one then carries out the familiar step-by-step process of determining an approximate solution at points nh, n = 1, 2, ...

Multistep methods have also been proposed [4, 6], but in the interests of simplicity we have restricted ourselves to one-step methods in this preliminary study. In Section 2 we look at some stability questions for Euler's method and in Section 3 we consider possible advantages of using a modified Euler's method.

2. STABILITY OF EULER'S METHOD

In a recent paper, [1], Cryer and Tavernini discussed an Euler's-method approach to the computation of solutions to Volterra functional differential equations. Under very general hypotheses they show that the method is stable in the sense that the errors are bounded. Just as for ordinary differential equations, however, one would expect that a stronger stability condition is required in order to compute valid solutions to a functional differential equation with a decreasing exponential solution. We here investigate this question in a particularly simple context.

Since it is impossible to investigate the stability properties of a method as it would apply to any equation, it is common in studying methods for ordinary differential equations to introduce the notion of a "test equation"--a simple equation for which the method is easily analyzed to yield information on how the method will work for a large class of "similar" equations. The standardly used test equation for ordinary differential equations is

$$(3) \quad y'(x) + ay = 0 \quad (a>0)$$

with initial condition $y(0) = 1$. Nothing this simple will provide a valid test for an integro-differential equation method, since an equation involving both an integral and a derivative is essentially analogous to a second-order differential equation. With this in mind, we decided on the test equation

$$(4) \quad \begin{aligned} & y' + ay + (a^2/4) \int_0^x y(t)\,dt = 0 \\ & y(0) = 1. \end{aligned}$$

For $a>0$, this has the solution

(5) $y = \exp(-ax/2)\,[1 - ax/2]$.

Although this is not the simple decreasing exponential obtained for (3), it does have the desired testing behavior of approaching zero asymptotically as $x \to +\infty$.

To solve (4) numerically, we define $x_n = nh$ $(n = 0, 1, \ldots)$ and $y(x_n) \equiv y_n$. Using Euler's method together with the trapezoidal rule for the quadrature, we then obtain the following numerical approximation to (4).

$$y_1 = (1 - ah)y_0$$

(6)
$$y_2 = \delta y_1 - (\gamma/2)y_0$$

$$y_{n+1} = \delta y_n - \gamma \sum_{j=1}^{n-1} y_j - (\gamma/2)\, y_0 \qquad (n>1)$$

where $\delta = 1 - ah - a^2h^2/8$

and $\gamma = a^2h^2/4$.

In order to study error propagation, we assume that y_0 is exact, and that an error E_1 is introduced into y_1. This error then propagates according to

(7)
$$E_2 = \delta E_1,$$

$$E_{n+1} = \delta E_n - \gamma \sum_{j=1}^{n-1} E_j.$$

Letting $\lambda \equiv ah$, one defines the region of absolute stability of the method as the set of complex values of λ for which $E_n \to 0$ as $n \to \infty$.

Since the order of the difference equation (7) changes with n, we use an indirect approach to determine the behavior of E_n. Define the accumulated error

$$T_n = \sum_{j=1}^{n} E_j .$$

Then, from (7), T_n satisfies the simple 3-term recursion

(8) $$T_{n+1} = (1+\delta)\, T_n - (\gamma+\delta)\, T_{n-1} \qquad (n \geq 1),$$

with $T_0 = 0$, $T_1 = E_1$.

The solution to equation (5) is

(9) $$T_n = \frac{E_1}{[(1+\delta)^2 - 4(\gamma+\delta)]^{1/2}} \left\{ q_1^n - q_2^n \right\},$$

where

(10) $$\begin{aligned} q_{1,2} &= 1/2 \left\{ (1+\delta) \pm [(1+\delta)^2 - 4(\gamma+\delta)]^{1/2} \right\} \\ &= 1 - (\lambda/2) - (\lambda^2/16) \pm (\lambda/4)[\lambda+\lambda^2/16]^{1/2}. \end{aligned}$$

It is clear that if both $|q_1|$ and $|q_2|$ are less than one, then $T_n \to 0$ as $n \to \infty$ and E_n must approach zero also. If we now write $E_n = T_n - T_{n-1}$ in terms of q_1 and q_2 and consider the various possibilities, it becomes clear that the conditions $|q_1|<1$, $|q_2|<1$ are also necessary for $E_n \to 0$. Thus the region of absolute stability is the domain in which both $|q_1|$ and $|q_2|$ are less than one. Looking at real values of λ, we see that the condition for this is $0<\lambda<2$. Computer trials for λ in the vicinity of 2 confirm that $\lambda = 2$ is the dividing point between error growth and

error decay. To determine the situation for complex λ, we have computed $|q_1|$, $|q_2|$ on a grid of values in the complex plane. The region on which they are both less than one is only slightly distorted from being the disk $|\lambda - 1| < 1$ (See the Figure).

For comparison, recall that the region of absolute stability for Euler's method applied to the ordinary differential equation (3) is precisely the disk $|\lambda-1|<1$, where again $\lambda = ah$. Intuitively, one might feel that addition of an integral operator term and the accompanying summing of errors (see (7)) would have a serious effect on the region of absolute stability. Our analysis shows that this is not the case.

A very closely related question is whether the numerical solution y_n approaches zero as $n \to \infty$ as it ought. This question may be investigated by a very similar procedure. Define

(11) $$Y_n = \sum_{j=0}^{n} y_j .$$

Then, using (6) with $y_o = 1$, we obtain

(12) $$Y_{n+1} = (1+\delta)Y_n - (\gamma+\delta)Y_{n-1} + (\gamma/2) \qquad (n \geq 1)$$

with initial conditions $Y_o = 1$, $Y_1 = 2 - \lambda$. This is an inhomogeneous difference equation with solution

(13) $$Y_n = \frac{1}{2(q_1-q_2)} \{(3-2\lambda-q_2)q_1{}^n - (3-2\lambda-q_1)q_2{}^n\} + 1/2,$$

where q_1 and q_2 are given by (10). If both $|q_1|$ and $|q_2|$ are less than one, $Y_n \to 1/2$ as $n \to \infty$, and the convergence of the series $\sum_{j=o}^{\infty} y_j$ implies that $y_j \to o$ as $j \to \infty$. Thus we see that, as commonly

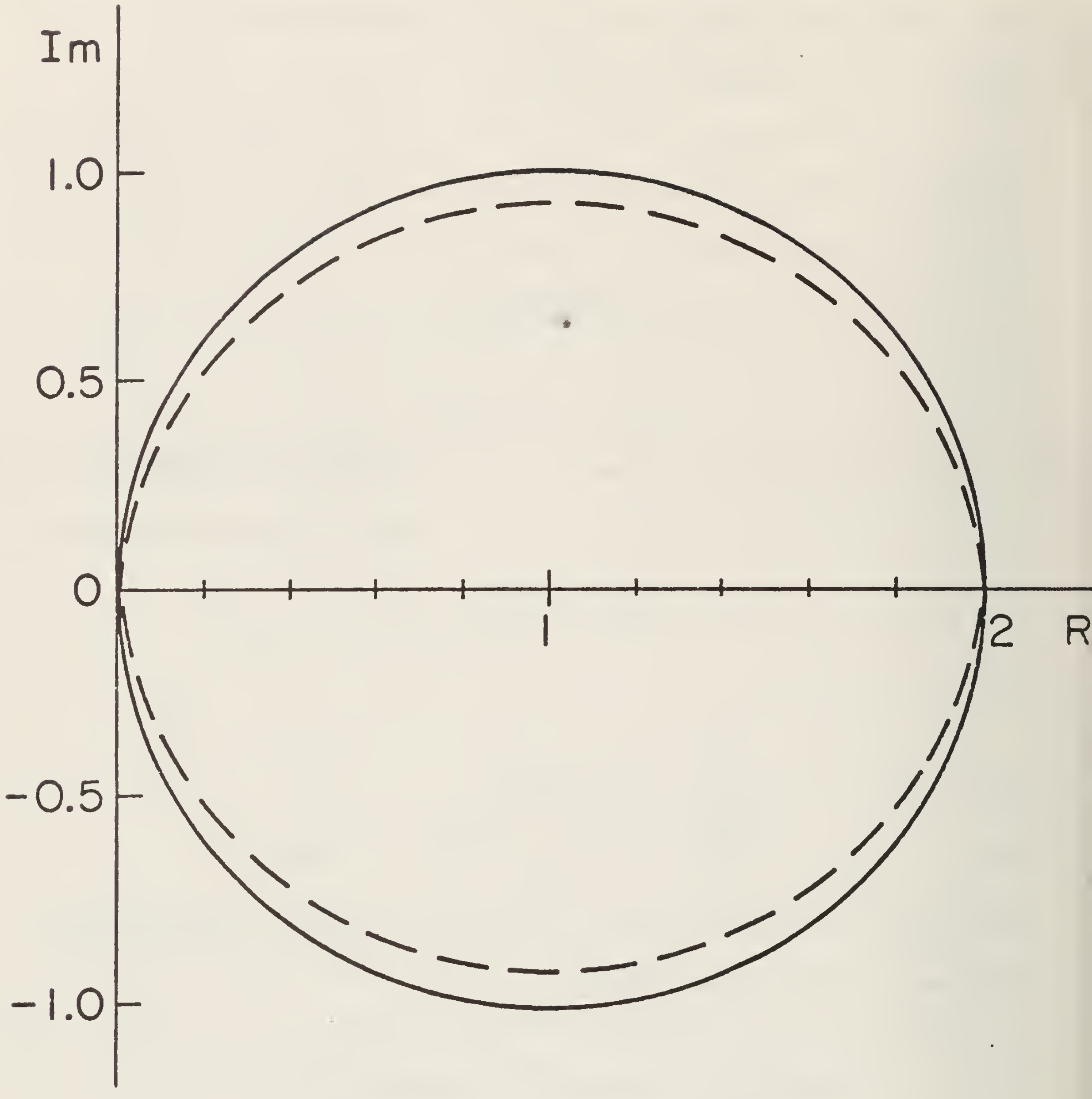

Figure. The broken line denotes the boundary of the region of absolute stability in the complex λ plane. For comparison, the solid line is the circle $|\lambda - 1| = 1$.

occurs in the case of ordinary differential equations, the numerical solution has the proper asymptotic behavior within the region of absolute stability.

In summary, it appears that the simple Euler's method is perhaps not as useful as a universal method as Cryer and Tavernini seem to suggest. The lack of absolute stability in some situations is a serious problem. Furthermore, there seems to be no straightforward way to determine a parameter λ which, for a general integro-differential equation, plays roughly the same role as the ah in the test equation. That is, there is much more work to be done on prediction of whether Euler's method is likely to be absolutely stable for a particular equation and a particular step size h.

3. GENERALIZED EULER METHODS

In this section we briefly consider the class of one-step methods which may be written

$$\frac{y_{n+1} - y_n}{h} = (1-\mu)\, F_{n+1} + \mu F_n, \tag{14}$$

where $0\leq\mu\leq 1$ and F_n represents the right side of (1) evaluated at x_n. (When F involves an integral, the "evaluation" involves an approximate quadrature, of course.) For $\mu = 1$ we have the simple Euler's method discussed in the last section. When $\mu<1$ the method becomes implicit, but the increase in computational complexity may be offset by improved stability properties (i.e. stability for larger values of h). In the case of ordinary differential equations this is certainly true. Applying (14) to the test equation (3) it is a trivial exercise to show that the method is "A-stable" (absolutely stable for all λ such that $\mathrm{Re}\lambda>0$) whenever $0\leq\mu<\frac{1}{2}$.

It seems very reasonable that this result will also extend to integro-differential equations. In an attempt to do this, the approach of Section 2, with the same test equation, has been applied to the generalized methods. One readily obtains a stability condition of the form $|q_1|$, $|q_2|<1$; the difficulty arises in trying to determine for what values of λ these conditions hold. Recall that in the last section the boundary of the domain of absolute stability was found by computing values of $|q_1|$, $|q_2|$. Such a computational approach has no hope of demonstrating absolute stability for half of the complex plane. Some other method must be devised. When λ is real, however, it is possible

to show analytically that absolute stability holds for all $\lambda>0$ when $0\le\mu<\frac{1}{2}$.

An advantage of having a class of methods depending upon a parameter (μ) is that this parameter may sometimes be adjusted so as to improve the numerical solution in some sense. Thus Liniger and Willoughby [5] have used the free parameter μ in order to fit the numerical solution to exponentially-decreasing behavior and thus obtain improved solutions to "stiff" ordinary differential equations.

We have done some experimentation on this sort of idea for the case of integro-differential equations. A difficulty is that the solution (5) to the test equation (4) is initially almost a straight line, the exponential dominance appearing later. Hence it is hard to see how a μ might be chosen to mimic the step-by-step behavior of the true solution throughout. Believing it of overriding importance to begin with a good numerical solution, we have opted to use μ to attempt to get an exact solution at the point $x_1 = h$.

Evaluating the exact solution (5) at h we get

$$y(h) = (1-\lambda/2)\exp(-\lambda/2), \tag{15}$$

while the numerical method (again with the trapezoidal rule used for the quadrature) yields

$$y_1 = \frac{(1-\lambda\mu-\lambda^2(1-\mu)/8)}{1+\lambda(1-\mu)+\lambda^2(1-\mu)/8}. \tag{16}$$

Equating the expressions in (15) and (16), we find that

$$\frac{(1-\lambda/2)\exp(-\lambda/2) - 1 + \lambda}{\lambda\{(1-\lambda/8) - (1-\lambda/2)(1+\lambda/8)\exp(-\lambda/2)\}} = 1-\mu \tag{17}$$

As λ increases, μ rapidly decreases to zero (when λ is about 4) and turns negative. Hence only a very small range of λ values produce meaningful values of the parameter μ from equation (17). There seems, therefore, to be very little value in pursuing this approach. It does appear to indicate, however, that if some member of the class of generalized Euler formulas (14) is to be chosen for routine use, the best choice might well be $\mu = 0$ (backward Euler), both for initial accuracy and long-term stability. It should be noted that all members of the class should have $O(h^2)$ local truncation error and thus should be first order. (See [1] for a detailed error analysis for Euler's method ($\mu = 1$).) It might well be worthwhile to develop higher order single-step methods for integro-differential equations.

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